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Optimal cubature formulas on compact homogeneous manifolds

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Abstract

We find lower bounds for the rate of convergence of optimal cubature formulas on sets of differentiable functions on compact homogeneous manifolds of rank I or two-point homogeneous spaces. It is shown that these lower bounds are sharp in the power scale in the case of $\mathbb{S}^2$, the unit sphere in $\mathbb{R}^3$.

Keywords: Reconstruction; Data points; Polynomial; Volume

1. Introduction

Let (Ω, Σ, η) be a measure space, where Ω is a compact domain in $\mathbb{R}^d$, and $\mathcal{K} \in C(\Omega)$ be a given set of real continuous functions, $f : \Omega \to \mathbb{R}$. Let $\{x_1, \ldots, x_n\} \subset \Omega$ be a fixed set of points. It is natural to approximate the integral

$$\int_\Omega f\, d\eta$$

by a cubature formula

$$\sum_{k=1}^{n} \alpha_k f(x_k),$$

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where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ and to minimize the error of approximation

$$\kappa_n(\mathcal{K}) := \inf_{\{x_1,\dots,x_n\}\subset\Omega} \inf_{\{\alpha_1,\dots,\alpha_n\}\subset\mathbb{R}} \sup_{f\in\mathcal{K}} \left| \int_{\Omega} f\, d\eta - \sum_{k=1}^{n} \alpha_k f(x_k) \right|. \tag{1}$$

The theory of cubature formulas has a long history and the first references are coming from the unmemorable times. In the modern epoch simple cubature (quadrature) formulas have been constructed by Kepler and Torricelli (1664), Simpson (1743), Newton and Cotes (1722). Different important methods of computing of integrals have been developed by Lagrange, Chebyshev, Bernstein, Krylov, Nikol'skij, Sobolev and many others. In the one-dimensional case, on $\mathbb{S}^1$, the unit circle, it is known that the formula of rectangles

$$\int_{\mathbb{S}^1} f(x)\, dx \approx \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{2\pi k}{n}\right)$$

is an optimal on Sobolev classes $W_\infty^r(\mathbb{S}^1) = \{f \mid f^{(r)} \in U_\infty(\mathbb{S}^1)\}$, where $U_\infty(\mathbb{S}^1) = \{\phi \mid \|\phi\|_\infty \leqslant 1\}$ and $r \in \mathbb{N}$. Remark that an analogous result is unknown for fractional values of $r > 0$.

Observe that the extremal problem (1) and their discrete analogs are in spirit of the classical Kolmogorov n-widths of finite-dimensional sets. This range of problems has been extensively studied by Tikhomorov, Makovoz, Kashin, Gluskin and others (see [17] for more details and references).

The problem of numerical integration over the surface of the unit sphere $\mathbb{S}^d$ in $\mathbb{R}^{d+1}$, $d \geqslant 2$, is one of the most important in Numerical Analysis and Applications. The theory of functions on $\mathbb{S}^2$ has been initiated in the eighteenth century in works of Laplace and Legendre when the first cubature formulas appeared. Consequently, the problem on an optimal cubature formula on $\mathbb{S}^2$ (in general, $\mathbb{S}^d$, $d \geqslant 2$) remains open since that time. Therefore, the problem on the best cubature formula on $\mathbb{S}^2$ or (in general) on compact Riemannian manifolds $\mathbb{M}^d$ it is natural to call the *Laplace–Legendre* problem.

A fundamental problem in this area is connected with an optimal distribution of data points $x_1, \dots, x_n$ and finding an optimal coefficients $\alpha_1, \dots, \alpha_n$ to approximate "well" the integral. Even in the case of $\mathbb{S}^2$, the two-dimensional sphere in $\mathbb{R}^3$, it is not possible to construct, in general, an equidistributed set of data points since there are finitely many polyhedral groups. Different attempts to find sets of points on the sphere which imitate the role of the roots of unity on the unit circle usually led to deep problems of the Geometry of Numbers, Theory of Potential, etc., and usually these approaches give just a measure of a uniform distribution like cup discrepancy or minimum energy configurations.

We consider here optimal cubature formulas for the Sobolev classes $W_\infty^r(\mathbb{M}^d) \subset C(\mathbb{M}^d)$ on a compact two-point homogeneous manifold $\mathbb{M}^d$ defined in Section 2.

The respective extremal problem can be formulated as following. Let $f \in W_\infty^r(\mathbb{M}^d)$ (see Section 2 for the definitions) and $\{x_1, \dots, x_n\} \subset \mathbb{M}^d$. Consider an information operator $T_n \in \mathcal{L}(C(\mathbb{M}^d), \mathbb{R}^n)$,

$$T_n : C\big(\mathbb{M}^d\big) \to \mathbb{R}^n,$$
$$f(\cdot) \mapsto \big(f(x_1), \dots, f(x_n)\big).$$

Let

$$\varrho\big(f(x_1),\dots,f(x_n)\big)=\varrho\circ T_n f$$

be a given function, $\varrho:\mathbb{R}^n\to\mathbb{R}$ (a recovery operator). Consider the extremal problem

$$\kappa_n\big(W^r_\infty\big(\mathbb{M}^d\big),\mathcal{R}\big):=\inf_{\{x_1,\dots,x_n\}\subset\mathbb{M}^d}\ \inf_{\varrho\in\mathcal{R}}\ \sup_{f\in W^r_\infty(\mathbb{M}^d)}\left|\int\limits_{\mathbb{M}^d} f\,d\nu-\varrho\circ T_n f\right|, \tag{2}$$

where $\mathcal{R}$ is a given class of functions $\varrho:\mathbb{R}^n\to\mathbb{R}$ and $d\nu$ is the invariant normalized measure on $\mathbb{M}^d$. We shall write $\kappa_n(W^r_\infty(\mathbb{M}^d))$ instead of $\kappa_n(W^r_\infty(\mathbb{M}^d),\mathcal{R})$ if $\mathcal{R}=\mathbb{R}^{\mathbb{R}^n}$ (the set of all functions $\varrho:\mathbb{R}^n\to\mathbb{R}$). In particular, if $\mathcal{R}$ is the set of linear functionals ϱ over $\mathbb{R}^n$ then for some $\alpha_1,\dots,\alpha_n$,

$$\varrho\circ T_n f=\sum_{k=1}^n\alpha_k f(x_k),$$

and (2) reduces to the Laplace–Legendre problem

$$\kappa'_n\big(W^r_\infty\big(\mathbb{M}^d\big)\big):=\inf_{\{x_1,\dots,x_n\}\subset\mathbb{M}^d}\ \inf_{\{\alpha_1,\dots,\alpha_n\}\subset\mathbb{R}}\ \sup_{f\in W^r_\infty(\mathbb{M}^d)}\left|\int\limits_{\mathbb{M}^d} f\,d\nu-\sum_{k=1}^n\alpha_k f(x_k)\right|. \tag{3}$$

The problem of construction of optimal cubature formulas splits into two parts, finding of a lower bond in (2) or (3) and obtainment of respective upper bounds. The lower bounds are of independent interest because they allow us to compare and classify a wide range of cubature formulas. In this article we develop a new method of obtainment of lower bounds for $\kappa_n(W^r_\infty(\mathbb{M}^d))$, $r>0$ on general compact two-point homogeneous manifolds $\mathbb{M}^d$ which are sharp in the power scale in the case of $\mathbb{S}^2$.

On the first step we apply the result of Smolyak [1] to reduce the problem (2) to the linear case (3).

Then, to find respective lower bounds for the rate of convergence of a cubature formula on Sobolev classes $W^r_\infty(\mathbb{M}^d)$ we consider the set $W^r_\infty(\mathbb{M}^d)\cap\ker T_n\cap\mathcal{T}_M$, where $\mathcal{T}_M$ is the set of polynomials of order $\leqslant M$ on $\mathbb{M}^d$ and $\dim\mathcal{T}_M=m\geqslant n$.

Then, applying Bernstein's inequality we reduce the problem to the consideration of the set $m^{-r/d}U_\infty(\mathbb{M}^d)\cap\mathcal{T}_M$, where $U_\infty(\mathbb{M}^d)$ is the unit ball in $L_\infty(\mathbb{M}^d)$.

Finally, we need to find a polynomial γ, $\deg\gamma\leqslant CM$, where $C>0$ is an absolute constant, such that $T_n\gamma=0$, $\|\gamma\|_\infty=1$ and the value $\int_{\mathbb{M}^d}\gamma\,d\nu$ is sufficiently big. Even in the case of the circle, $\mathbb{S}^1$, it is very difficult to construct a polynomial with such properties. We show the existence of such objects using methods of geometry of Banach spaces. Remark that in applications we have a little information concerning special convex bodies in $\mathbb{R}^n$ which are connected with the structure of a fixed system of spherical harmonics on $\mathbb{M}^d$. This is a source of fundamental difficulties which occur if we try to apply the results of geometry of Banach spaces to various open problems in different spaces of functions. A useful tool in this range of problems are Levy means defined in Section 3. We employ estimates of Levy means in combination with the Bieberbach inequality and the Brunn–Minkowski theorem. Note that estimates of Levy means connected with different orthonormal systems have been obtained in [10–15].

The results we derive are apparently new even in the one-dimensional case and the method's possibilities are not confined to the statements proved but can be applied in studying more general problems.

We use several universal constants which enter into the estimates. These positive constants are mostly denoted by the letter C. We did not carefully distinguish between the different constants, neither did we try to get good estimates for them. The same letter will be used to denote different universal constants in different parts of the article. For ease of notation we will write $a_n \ll b_n$ for two sequences, if $a_n \leqslant C b_n$ for $n \in \mathbb{N}$ and $a_n \asymp b_n$, if $C_1 b_n \leqslant a_n \leqslant C_2 b_n$ for all $n \in \mathbb{N}$ and some constants C, C_1 and C_2.

2. Harmonic analysis

A Riemannian manifold is two-point homogeneous if for any set of four points x_1, y_1, x_2, y_2 with $d(x_1, y_1) = d(x_2, y_2)$, d being the Riemannian metric on $\mathbb{M}^d$, there exists $\phi \in \mathcal{G}$ such that $\phi(x_1) = x_2$ and $\phi(y_1) = y_2$. A complete classification of the two-point homogeneous spaces was given in [18]. For information on this classification see, e.g., [4,6–9] They are: the spheres $\mathbb{S}^d$, $d = 1, 2, 3, \dots$; the real projective spaces $P^d(\mathbb{R})$, $d = 2, 3, 4, \dots$; the complex projective spaces $P^d(\mathbb{C})$, $l = d/2$, $d = 4, 6, 8, \dots$; the quaternionic projective spaces $P^d(\mathbb{H})$, $d = 8, 12, \dots$; the Cayley elliptic plane $P^{16}(\mathrm{Cay})$. The superscripts here denote the dimension over the reals of the underlying manifolds $\mathbb{M}^d$. Each $\mathbb{M}^d$ can be considered as the orbit space of some compact subgroup $\mathcal{H}$ of the orthogonal group $\mathcal{G}$, that is $\mathbb{M}^d = \mathcal{G}/\mathcal{H}$. Let $\pi : \mathcal{G} \to \mathcal{G}/\mathcal{H}$ be the natural mapping and $\mathbf{e}$ be the identity of $\mathcal{G}$. On any such manifold there is an invariant Riemannian metric $d(\cdot,\cdot)$, and an invariant Haar measure $d\nu$. The point $\mathbf{o} = \pi(\mathbf{e})$ which is invariant under all motions of $\mathcal{H}$ is called the pole of $\mathbb{M}^d$. A function $Z = Z^{\mathbf{o}} : \mathbb{M}^d \to \mathbb{R}$ is called zonal with the pole $\mathbf{o} \in \mathbb{M}^d$ if $Z^{\mathbf{o}}(h^{-1}\cdot) = Z^{\mathbf{o}}(\cdot)$ for any $h \in \mathcal{H}$.

Consider a two-point homogeneous space $\mathbb{M}^d$. Let g be its metric tensor, ν its normalized volume element and Δ its Laplace–Beltrami operator. In local coordinates x_l, $1 \leqslant l \leqslant d$,

$$\Delta = -(\bar{g})^{-1/2} \sum_k \frac{\partial}{\partial x_k} \left(\sum_j g^{jk} (\bar{g})^{1/2} \frac{\partial}{\partial x_j} \right),$$

where $g_{jk} := g(\partial/x_j, \partial/x_k)$, $\bar{g} := |\det(g_{jk})|$, and $(g^{jk}) := (g_{jk})^{-1}$. It is well known that Δ is an elliptic, self-adjoint, invariant under isometries, second order operator. The eigenvalues θ_k, $k \geqslant 0$ of Δ are discrete, nonnegative and form an increasing sequence $0 \leqslant \theta_0 \leqslant \theta_1 \leqslant \cdots \leqslant \theta_n \leqslant \cdots$ with $+\infty$ the only accumulation point and $\theta_k \asymp k^2$. Corresponding eigenspaces H_k, $k \geqslant 0$ are finite-dimensional, $d_k := \dim \mathrm{H}_k$, orthogonal and $L_2(\mathbb{M}^d, \nu) = \bigoplus_{k=0}^{\infty} \mathrm{H}_k$. It is possible to show that $d_k := \dim \mathrm{H}_k \asymp k^{d-1}$, $m := \dim \mathcal{T}_M \asymp M^d$, where $\mathcal{T}_M := \bigoplus_{k=0}^{M} \mathrm{H}_k$. Let us fix a real orthonormal basis $\{Y_l^k\}_{l=1}^{d_k}$ of H_k.

For an arbitrary $\phi \in L_p(\mathbb{M}^d)$, $1 \leqslant p \leqslant \infty$ with the formal Fourier series

$$\phi \sim c_0 + \sum_{k \in \mathbb{N}} \sum_{l=1}^{d_k} c_{k,l}(\phi) Y_l^k, \quad c_{k,l}(\phi) = \int_{\mathbb{M}^d} \phi Y_l^k \, d\nu,$$

the rth fractional integral ϕ_r, $r > 0$, is defined as

$$\phi_r \sim c_0 + \sum_{k\in\mathbb{N}} \theta_k^{-r/2} \sum_{l=1}^{d_k} c_{k,l}(\phi) Y_l^k, \quad c_0 \in \mathbb{R}. \tag{4}$$

The last equation defines the operator of fractional integration $\phi_r = I_r\phi$. The function $\phi^{(r)} \in L_p(\mathbb{M}^d)$, $1 \leqslant p \leqslant \infty$, is called the rth fractional derivative of ϕ if

$$\phi^{(r)} \sim \sum_{k\in\mathbb{N}} \theta_k^{r/2} \sum_{l=1}^{d_k} c_{k,l}(\phi) Y_l^k.$$

Sobolev classes $W_p^r(\mathbb{M}^d)$ are defined as sets of functions with formal Fourier expansions (4) where $\|\phi\|_p \leqslant 1$ and $\int_{\mathbb{M}^d} \phi\, d\nu = 0$ (see [2] for details).

3. The results

The main result of this article is the following

Theorem 1. *For any $r > 0$ and $\varepsilon > 0$ we have*

$$\kappa_n\big(W_\infty^r\big(\mathbb{M}^d\big)\big) = \kappa_n'\big(W_\infty^r\big(\mathbb{M}^d\big)\big) \geqslant C_\varepsilon n^{-r/d} (\log n)^{-(1+\varepsilon)},$$

where C_ϵ depends just on $\varepsilon > 0$.

Proof. It was discovered by S.A. Smolyak and then published in [1] that for the recovery of linear functionals using linear information it is sufficient to use linear methods. More precisely, let $\mathcal{K}$ be a convex centrally symmetric subset of a Banach space X, $T_n : X \to \mathbb{R}^n$,

$$T_n x = \big(\langle x_1', x\rangle, \dots, \langle x_n', x\rangle\big), \quad x_k' \in X',\ 1 \leqslant k \leqslant n,$$

and ξ a linear functional ($\xi(x) = \langle x_0', x\rangle$, $x_0' \in X'$). Then there is an optimal linear method of recovery, that is

$$\inf_{\varrho\in\mathcal{R}} \sup_{x\in\mathcal{K}} \big|\langle x_0', x\rangle - \varrho \circ T_n x\big| = \inf_{(a_1,\dots,a_n)\in\mathbb{R}^n} \sup_{x\in\mathcal{K}} \left|\langle x_0', x\rangle - \sum_{k=1}^n a_k \langle x_k', x\rangle\right|,$$

where $\mathcal{R} = \mathbb{R}^{\mathbb{R}^n}$ is the set of all functions $\varrho : \mathbb{R}^n \to \mathbb{R}$. Remark that the Sobolev classes $W_\infty^r(\mathbb{M}^d)$ are convex and centrally symmetric. Hence, from (2) and (3) we get

$$\kappa_n\big(W_\infty^r\big(\mathbb{M}^d\big)\big) = \kappa_n'\big(W_\infty^r\big(\mathbb{M}^d\big)\big) \geqslant \inf_{T_n} \sup_{f\in W_\infty^r(\mathbb{M}^d)\cap\ker T_n} \left|\int_{\mathbb{M}^d} f\, d\nu\right|,$$

where $T_n f = (f(x_1), \dots, f(x_n))$. Clearly, $\operatorname{codim}\ker T_n \leqslant n$. In what follows we show that for any $\{x_1, \dots, x_n\} \subset \mathbb{M}^d$ there is a polynomial

$$\gamma_{(x_1,\dots,x_n)}^* \in W_\infty\big(\mathbb{M}^d\big) \cap \mathcal{T}_{2M} \cap \ker T_n, \quad \dim \mathcal{T}_M = m \geqslant n$$

such that for any $\varepsilon > 0$

$$\int_{\mathbb{M}^d} \gamma^*_{(x_1,\dots,x_n)}\, d\nu \geqslant C_\epsilon n^{-r/d} (\log n)^{-(1+\varepsilon)}, \tag{5}$$

where $C_\epsilon > 0$ depends just on ϵ. Let $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m$, $\beta = (\beta_1, \dots, \beta_m) \in \mathbb{R}^m$ and $[\alpha, \beta] = \sum_{k=1}^m \alpha_k \beta_k$. Let $\|\alpha\|_{(2)} = [\alpha, \alpha]^{1/2}$ be the Euclidean norm on $\mathbb{R}^m$, $\mathbb{S}^{m-1} = \{\alpha \in \mathbb{R}^m \mid \|\alpha\|_{(2)} = 1\}$ be the unit sphere in $\mathbb{R}^m$, $B^m_{(2)} = \{\alpha \in \mathbb{R}^n \mid \|\alpha\|_{(2)} \leqslant 1\}$ be the unit ball in $\mathbb{R}^m$ and Vol_m be the standard m-dimensional volume of subsets in $\mathbb{R}^m$. Let us fix a norm $\|\cdot\|$ on $\mathbb{R}^m$ and denote by E the Banach space $E = (\mathbb{R}^m, \|\cdot\|)$ with the unit ball $B_E = V$. The Levy mean M_V is defined by

$$M_V = M\big(\mathbb{R}^m, \|\cdot\|\big) = \int_{\mathbb{S}^{m-1}} \|\alpha\|\, d\mu(\alpha),$$

where $d\mu$ is the invariant normalized measure on $\mathbb{S}^{m-1}$. Consider the coordinate isomorphism

$$J : \mathbb{R}^m \to \mathcal{T}_M, \quad \dim \mathcal{T}_M = m,$$

that assigns to $\alpha = (\alpha_1, \dots, \alpha_m) = (\alpha_{k,l},\ 0 \leqslant k \leqslant M,\ 1 \leqslant l \leqslant d_k) \in \mathbb{R}^m$ the function $J\alpha = \xi^\alpha = \sum_{k=0}^M \sum_{l=1}^{d_k} \alpha_{k,l} Y^k_l \in \mathcal{T}_M$. The definition $\|\alpha\|_{(\infty)} = \|\xi^\alpha\|_\infty$ induces a norm on $\mathbb{R}^m$. Put

$$B^m_{(\infty)} := \big\{\alpha \bigm| \alpha \in \mathbb{R}^m,\ \|\alpha\|_{(\infty)} \leqslant 1\big\},$$

then $B^m_\infty := J B^m_{(\infty)}$. The following estimate plays an important role in our method [15]

$$M\big(\mathbb{R}^m, \|\cdot\|_{(\infty)}\big) \leqslant C (\log m)^{1/2}. \tag{6}$$

Let V be a convex centrally symmetric body in $\mathbb{R}^m$ which is the unit ball in $E(\mathbb{R}^m, \|\cdot\|)$. A direct calculation (see, e.g., [16]) shows that

$$\left(\frac{\mathrm{Vol}_m(V)}{\mathrm{Vol}_m(B^m_{(2)})}\right) = \int_{\mathbb{S}^{m-1}} \|\alpha\|^{-m}\, d\mu(\alpha)$$

and by convexity we get

$$\left(\int_{\mathbb{S}^{m-1}} \|\alpha\|^{-m}\, d\mu(\alpha)\right)^{1/m} \geqslant \left(\int_{\mathbb{S}^{m-1}} \|\alpha\|\, d\mu(\alpha)\right)^{-1} = M_V^{-1}.$$

Hence, by (6) it follows that

$$\mathrm{Vol}_m\big(B^m_{(\infty)}\big) \geqslant C^m (\log m)^{-m/2} \mathrm{Vol}_m\big(B^m_{(2)}\big). \tag{7}$$

Let $L_s \subset \mathbb{R}^m$ be any s-dimensional subspace, $(L_s)^\perp$ be the orthogonal complement of L_s and $P_{(L_s)^\perp}(B^m_{(\infty)})$ be the orthogonal projection of $B^m_{(\infty)}$ onto $(L_s)^\perp$. It is easy to check that $B^m_{(\infty)} \subset B^m_{(2)}$ and therefore $P_{(L_s)^\perp}(B^m_{(\infty)}) \subset P_{(L_s)^\perp}(B^m_{(2)})$ and $\mathrm{Vol}_{m-s}(P_{(L_s)^\perp}(B^m_{(\infty)})) \leqslant \mathrm{Vol}_{m-s}(B^{m-s}_{(2)})$.

Hence,

$$\mathrm{Vol}_m\big(B^m_{(\infty)}\big) = \int\limits_{B^m_{(\infty)}} dx = \int\limits_{P_{(L_s)^\perp}(B^m_{(\infty)})} \mathrm{Vol}_s\big(B^m_{(\infty)} \cap (y + L_s)\big)\, dy.$$

Thus, involving standard arguments connected with the Brunn–Minkowski theorem we get $\mathrm{Vol}_s(B^m_{(\infty)} \cap (y + L_s)) \leqslant \mathrm{Vol}_s(B^m_{(\infty)} \cap (L_s))$ for any $y \in P_{(L_s)^\perp}(B^m_{(\infty)})$. Consequently,

$$\begin{aligned}\mathrm{Vol}_m\big(B^m_{(\infty)}\big) &\leqslant \mathrm{Vol}_s\big(B^m_{(\infty)} \cap L_s\big) \cdot \mathrm{Vol}_{m-s}\big(P_{(L_s)^\perp}\big(B^m_{(\infty)}\big)\big) \\ &\leqslant \mathrm{Vol}_s\big(B^m_{(\infty)} \cap L_s\big) \cdot \mathrm{Vol}_{m-s}\big(B^{m-s}_{(2)}\big). \qquad (8)\end{aligned}$$

Hence, by (7) and (8),

$$\mathrm{Vol}_s\big(B^m_{(\infty)} \cap L_s\big) \geqslant C^m (\log m)^{-m/2} \frac{\mathrm{Vol}_m(B^m_{(2)})}{\mathrm{Vol}_{m-s}(B^{m-s}_{(2)})}. \qquad (9)$$

Comparing (9) with the Bieberbach inequality [3, p. 93],

$$\mathrm{diam}(V) \geqslant 2\left(\frac{\mathrm{Vol}_s(V)}{\mathrm{Vol}_s(B^s_{(2)})}\right)^{1/s},$$

which is valid, in particular, for any convex centrally symmetric body $V \subset \mathbb{R}^s$ we get the lower bound for the diameter of the set $B^m_{(\infty)} \cap L_s$,

$$\mathrm{diam}\big(B^m_{(\infty)} \cap L_s\big) \geqslant C^{m/s} (\log n)^{-m/(2s)} \left(\frac{\mathrm{Vol}_m(B^m_{(2)})}{\mathrm{Vol}_{m-s}(B^{m-s}_{(2)})\mathrm{Vol}_s(B^s_{(2)})}\right)^{1/s}.$$

Recall that

$$\mathrm{Vol}_m\big(B^m_{(2)}\big) = \pi^{m/2} \Gamma^{-1}\left(\frac{m}{2} + 1\right)$$

and

$$\Gamma(z) = z^{z-1/2} e^{-z} (2\pi)^{1/2} \big(1 + O\big(z^{-1}\big)\big), \quad z \to \infty.$$

Hence,

$$\left(\frac{\mathrm{Vol}_m(B^m_{(2)})}{\mathrm{Vol}_{m-s}(B^{m-s}_{(2)})\mathrm{Vol}_s(B^s_{(2)})}\right)^{1/s} \asymp \left(\frac{m}{s}\right)^{-1/2} m^{-1/(2s)}, \\ m \to \infty, \quad s \to \infty, \quad s \leqslant m.$$

It means that for any $0 < \lambda < 1$ and $s = [\lambda m]$ we have

$$\mathrm{diam}\big(B^m_{(\infty)} \cap L_s\big) \geqslant C_\lambda (\log m)^{-1/(2\lambda)}$$

or for any fixed $\{x_1,\dots,x_n\}\subset\mathbb{M}^d$, $n=m-s$, there is a polynomial $t_m\in\mathcal{T}_M$, $\dim\mathcal{T}_M=m$, such that $t_m(x_k)=0$, $1\leqslant k\leqslant n$, $\|t_m\|_\infty=1$ and $\|t_m\|_2\geqslant C_\lambda(\log m)^{-1/(2\lambda)}$. Consider the polynomial $\gamma^*=\gamma^*_{(x_1,\dots,x_n)}=(\dim\mathcal{T}_{2M})^{-r/d}t_m^2$. Clearly, $\gamma^*\in\mathcal{T}_{2M}$, $\gamma^*\geqslant 0$, $\|\gamma^*\|_\infty=(\dim\mathcal{T}_{2M})^{-r/d}$, $\gamma^*(x_k)=0$, $1\leqslant k\leqslant n$ and

$$\int\limits_{\mathbb{M}^d}\gamma^*\,d\nu=(\dim\mathcal{T}_{2M})^{-r/d}\|t_m\|_2^2\geqslant C_\lambda(\dim\mathcal{T}_{2M})^{-r/d}(\log m)^{-1/\lambda}.$$

Applying Bernstein's inequality [5] we get $C_{r,d}(\dim\mathcal{T}_{2M})^{-r/d}U_\infty(\mathbb{M}^d)\cap\mathcal{T}_{2M}\subset W^r_\infty(\mathbb{M}^d)$ or $\gamma^*\in CW^r_\infty\cap\mathcal{T}_{2M}$, from which (5) follows. $\square$

Remark 1. Let $x=(\theta,\phi)\in[0,\pi]\times[0,2\pi)$, $b=2l$, $l\in\mathbb{N}$, $\theta_l=\pi l/(2b)$, $0\leqslant l\leqslant 2b-1$, $\phi_k=\pi k/b$, $0\leqslant k\leqslant 2b-1$ and

$$a_l^b=\frac{1}{2b^2}\sin\left(\frac{l\pi}{2b}\right)\sum_{s=0}^{b/2-1}\frac{1}{2s+1}\sin\left(\frac{(2s+1)l\pi}{2b}\right).$$

It has been shown in [14] that for any $r>0$

$$\kappa_n\big(W^r_\infty(\mathbb{S}^2)\big)=\kappa'_n\big(W^r_\infty(\mathbb{S}^2)\big)\leqslant\sup_{f\in W^r_\infty(\mathbb{S}^2)}\left|\int\limits_{\mathbb{S}^2}f(x)\,dx-\sum_{l=0}^{2b-1}a_l^b\sum_{k=0}^{2b-1}f(\theta_l,\phi_k)\right|\ll n^{-r/2},$$

where $n=\operatorname{Card}\{(\theta_l,\phi_k),\ 0\leqslant l\leqslant 2b-1,\ 0\leqslant k\leqslant 2b-1\}\asymp b^2$. From Theorem 1 it follows that

$$\kappa_n\big(W^r_\infty(\mathbb{S}^2)\big)\gg n^{-r/2}(\log n)^{-(1+\varepsilon)}$$

for any $\varepsilon>0$. Hence, the lower bound given by Theorem 1 is sharp in the power scale in the case of Sobolev's classes on $\mathbb{S}^2$.

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